Summary: Vector operations so far

$c\mathbf{v}$ = "vector parallel to $\mathbf{v}$ with length scaled by factor of c"

$\mathbf{a} + \mathbf{b}$ = "if $\mathbf{a}$ and $\mathbf{b}$ are drawn tail-to- head, then $\mathbf{a} + \mathbf{b}$ is the vector that goes from the tail of $\mathbf{a}$ to the head of $\mathbf{b}$" (resultant/combined force)

12.3 Dot Products *(new)*

If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$

Then we define the dot product by:

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product gives a number (scalar).

Entry Task: $\mathbf{a} = \langle 3, 1, 2 \rangle$

$\mathbf{b} = -\mathbf{i} + 6\mathbf{j} + 5\mathbf{k}$

$b = \langle -1, 6, 5 \rangle$

Compute $\mathbf{a} \cdot \mathbf{b}$

must use dot symbol!

$\langle 3, 1, 2 \rangle \cdot \langle -1, 6, 5 \rangle$

$(3)(-1) + (1)(6) + (2)(5)$

$-3 + 6 + 10 = \boxed{13}$

dot product

Basic fact list:

- Manipulation facts (*like regular multiplication*):
  - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
  - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
  - $\mathrm{c}(\mathbf{a} \cdot \mathbf{b}) = (\mathrm{c}\mathbf{a}) \cdot \mathbf{b} = \mathbf{a} \cdot (\mathrm{c}\mathbf{b})$
  - $(\mathbf{a} + \mathbf{b}) \cdot (\mathbf{a} + \mathbf{b}) = ???$

FOIL!

$$(\vec{a} \cdot \vec{a} + \vec{b} \cdot \vec{a} + \vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{b})$$

- Helpful side fact:
  - $\mathbf{a} \cdot \mathbf{a} = \mathrm{a}_1^2 + a_2^2 + a_3^2 = |\boldsymbol{a}|^2$

$a = \langle 3, 2, 4 \rangle$

$\langle 3, 2, 4 \rangle \cdot \langle 3, 2, 4 \rangle$

$3^2 + 2^2 + 4^2 =$

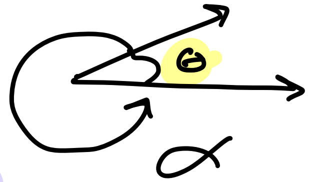
$(\text{magnitude})^2$

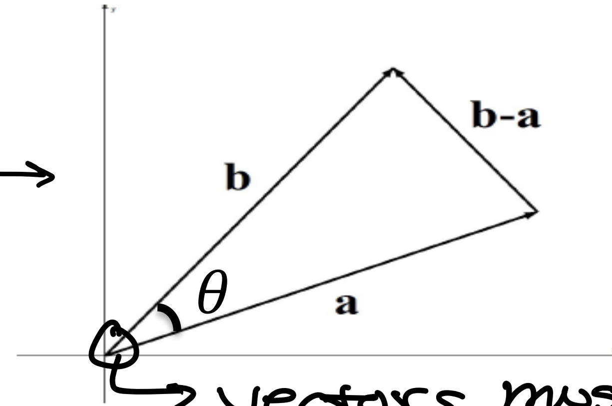
$|\vec{a}|^2 = \vec{a} \cdot \vec{a}$

The most important fact:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos(\theta),$$

where θ is the smallest angle between $\boldsymbol{a}$ and $\boldsymbol{b}$. $(0 \le \theta \le \pi)$





to apply

vectors must be tail to tail

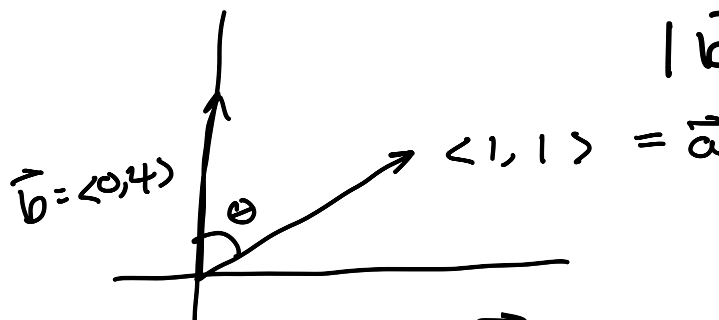
Something to think about

What does it mean when…

- the dot product is positive? acute
- the dot product is negative? obtuse
- the dot product is zero? perpendicular

$\vec{a}\cdot\vec{b} = 0+4 = 4 \qquad |\vec{a}| = \sqrt{2}$

$|\vec{b}| = 4$



$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$4 = 4\sqrt{2}\cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}} \quad \boxed{\theta = \pi/4}$

Proof (not required):

(1) By the Law of Cosines:

$$|\mathbf{b}-\mathbf{a}|^2 = |\boldsymbol{a}|^2 + |\boldsymbol{b}|^2 - 2|\mathbf{a}||\mathbf{b}|\cos(\theta)$$

The left-hand side expands to

$$\begin{aligned}|\mathbf{b}-\mathbf{a}|^2 &= (\mathbf{b}-\mathbf{a})\cdot(\mathbf{b}-\mathbf{a})\\ &= \mathbf{b}\cdot\mathbf{b} - 2\mathbf{a}\cdot\mathbf{b} + \mathbf{a}\cdot\mathbf{a}\\ &= |\mathbf{b}|^2 - 2\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2\end{aligned}$$

Subtracting $|\boldsymbol{a}|^2 + |\boldsymbol{b}|^2$ from both sides of (1) yields:

$$-2\mathbf{a}\cdot\mathbf{b} = -2|\mathbf{a}||\mathbf{b}|\cos(\theta).$$

Divide by -2 to get the result. (QED)

perpendicular

Most important consequence:
If **a** and **b** are orthogonal, then
$$\mathbf{a}\cdot\mathbf{b}=0.$$

Also: If **a** and **b** are parallel, then
$\mathbf{a}\cdot\mathbf{b} = |a||b|$ or $\mathbf{a}\cdot\mathbf{b} = -|a||b|$.

both parallel

$\langle 1, 1, 2\rangle = \vec{a}$

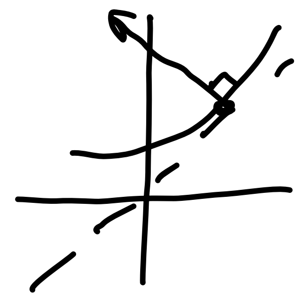
$\langle 3, 2, -5\rangle = \vec{b}$

$\langle -6, 4, 1\rangle = \vec{c}$

$\vec{a}\cdot\vec{c} = 0$

if multiples of each other = parallel

Example:
Find a vector that is orthogonal to the tangent line to $y = x^3 e^{(2x-2)}$ at $x = 1$.



slope of tangent line @ x=1

$$\frac{dy}{dx} = x^3 e^{(2x-2)}\cdot 2 + 3x^2 e^{(2x-2)}$$

$$\left.\frac{dy}{dx}\right|_{x=1} = 1(e^0)\cdot 2 + 3(1)e^0 = 5$$

$\vec{v} = \langle \Delta x, \Delta y\rangle \leftarrow$ slope=5 $= \frac{\Delta y}{\Delta x}$

$\vec{v} = \langle 1, 5\rangle \rightarrow$ parallel to tangent

$\vec{w} = \langle \Delta x, \Delta y\rangle = \boxed{\langle -5, 1\rangle \text{ or } \langle 5, -1\rangle}$

(flip and change 1 sign)

two vectors on top of each other are scaled versions of each other

Exploring Dot Products and Projections

Consider

$$\boldsymbol{u} = \langle 1, 2, 0 \rangle \text{ and } \boldsymbol{v} = \langle 0, 2, 3 \rangle.$$

(a) What is the dot product?

$(1)(0) + (2)(2) + (0)(3)$

$= \boxed{4}$

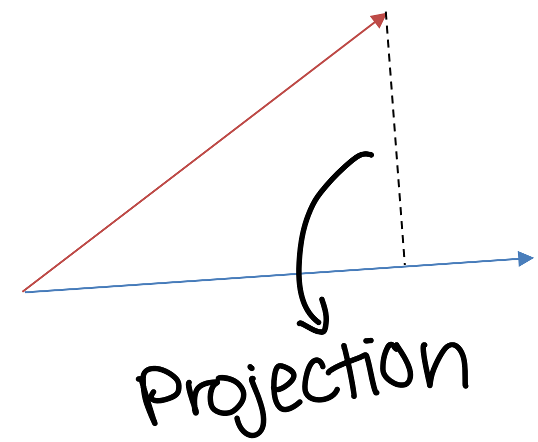
(b) Find the angle between the vectors.

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$

$4 = \left(\sqrt{1+2^2+0}\right)\left(\sqrt{0+4+9}\right) \cos\theta$

(c) Give me an example of a vector $\boldsymbol{w} = \langle a, b, c \rangle$ that is orthogonal to $\boldsymbol{u} = \langle 1, 2, 0 \rangle$?

(d) Now consider the right triangle you get by *projecting* ***v*** onto ***u***. What is the length of the adjacent side of that right triangle?

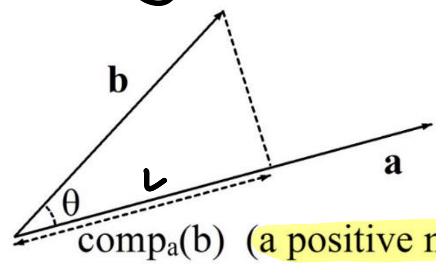


Visual: https://www.math3d.org/fSfgCqmvW thing projecting

Scalar/Component Projection

$$\underbrace{\mathrm{comp}}_{\#}{}_{\mathbf{a}}(\boldsymbol{b}) = \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}$$

onto



comp$_a$(b) (a positive number)

acute

comp$_a$(b) (a negative number)

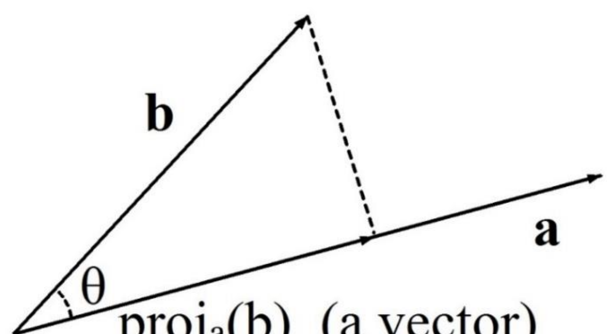
obtuse

$$\cos\theta = \frac{L}{|\vec{b}|} \qquad L = |\vec{b}|\cos\theta \qquad \vec{a}\cdot\vec{b} = |\vec{a}|\,|\vec{b}|\cos\theta$$

$$L = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$$

Vector Projection

$$\mathbf{proj}_{\mathbf{a}}(\boldsymbol{b}) = \overbrace{\frac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}|}}^{\text{scale}}\left(\overbrace{\frac{1}{|\boldsymbol{a}|}\boldsymbol{a}}^{\text{unit vector}}\right)$$
$$= \frac{\boldsymbol{a}\cdot\boldsymbol{b}}{\boldsymbol{a}\cdot\boldsymbol{a}}\ \boldsymbol{a}$$



proj$_a$(b) (a vector)

Visual: https://www.math3d.org/LDw4RPUy